THRESHOLDS AND RESONANCES OF SCHRÖDINGER OPERATORS ON A LATTICE (Mathematical Aspects of Quantum Fields and Related Topics)

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THRESHOLDS AND RESONANCES OF SCHRÖDINGER OPERATORS ON A LATTICE

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1 Discrete Schrödinger operators $H_{\lambda\mu}$ on lattice

This is the joint work with Z. Muminov and U. Kuljanov [2] which is a continuation of [3] where a discrete Schrödinger operator with single delta potential is considered. See also related paper [1]. In the present article we consider discrete Schrödinger operators with multi-delta potentials. Let $\mathbb{Z}^n$ be the n–dimensional lattice. The Hilbert space of ℓ^2 sequences on $\mathbb{Z}^n$ is denoted by $\ell^2(\mathbb{Z}^n)$, and we use $\ell^2_+(\mathbb{Z}^n)$ (resp. $\ell^2_-(\mathbb{Z}^n)$) to denote its subspace of all even (resp. all odd) functions. Let $T(y)$ be the shift operator by $y \in \mathbb{Z}^n$: $(T(y)f)(x) = f(x+y)$ for $f \in \ell^2(\mathbb{Z}^n)$ and $x \in \mathbb{Z}^n$. Let $\Delta = \frac{1}{2}\sum_{\substack{x\in\mathbb{Z}^n\\|x|=1}}(T(x)-T(0))$ be the discrete Laplacian on $\ell^2(\mathbb{Z}^n)$. Thus the discrete Schrödinger operator on $\ell^2(\mathbb{Z}^n)$ is defined by

$$\hat{H}_{\lambda\mu} = -\Delta - \widehat{V},$$

where the potential $\widehat{V}$ (Figure1) depends on two parameters $\lambda, \mu \in \mathbb{R}$ and satisfies

$$(\widehat{V}f)(x) = \begin{cases} \mu f(x), & \text{if} \quad x = 0 \\ \frac{\lambda}{2} f(x), & \text{if} \quad |x| = 1 \\ 0, & \text{if} \quad |x| > 1 \end{cases}$$

which awards $\hat{H}_{\lambda\mu}$ to be a bounded self-adjoint operator. A notation $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n = (-\pi,\pi]^n$ means the n-dimensional torus equipped with its Haar measure. We set $L^2(\mathbb{T}^n) = L^2$, and let L^2_+ (resp. L^2_-) denote the subspace of all even (resp. odd) functions of the Hilbert space L^2 of L^2-functions on $\mathbb{T}^n$. Let $\langle\cdot,\cdot\rangle$ mean the inner product on L^2. Let $\mathscr{F}$ be the standard Fourier transform $\mathscr{F}: L^2 \longrightarrow \ell^2(\mathbb{Z}^n)$ defined by $\mathscr{F}f(x) = (2\pi)^{-n}\int_{\mathbb{T}^n} f(\theta)e^{-ix\theta}d\theta$. Then the inverse Fourier transform is given by $\mathscr{F}^{-1}f(\theta) = \sum_{x\in\mathbb{Z}^n} f(x)e^{ix\theta}$. The Laplacian Δ in the momentum representation is defined as $\widehat{\Delta} = \mathscr{F}^{-1}\Delta\mathscr{F}$, and $\widehat{\Delta}$ acts as the multiplication operator: $(\widehat{\Delta}\hat{f})(p) = -E(p)\hat{f}(p)$, where $E(p)$ is given by $E(p) = \sum_{j=1}^n(1-\cos p_j)$. Set $H_0 = -\hat{\Delta}$. The operator $H_{\lambda\mu}$ acts L^2 as

$$H_{\lambda\mu} = H_0 - V,$$

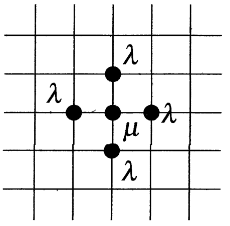


Figure 1: Potential V

and V is an integral operator of convolution type

$$(Vf)(p) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{T}^n} v(p-s) f(s) ds, \quad f \in L^2.$$

Here the kernel function is $v(p) = \frac{1}{(2\pi)^{n/2}} (\mu + \lambda \sum_{i=1}^{n} \cos p_i)$, and it allows the potential operator V to get the representation $V = V^{+}_{\lambda\mu} + V^{-}_{\lambda}$, where

$$V^{+}_{\lambda\mu} = \mu \langle \cdot, c_0 \rangle c_0 + \frac{\lambda}{2} \sum_{j=1}^{n} \langle \cdot, c_j \rangle c_j, \quad V^{-}_{\lambda} = \frac{\lambda}{2} \sum_{j=1}^{n} \langle \cdot, s_j \rangle s_j.$$

Here $\{c_0, c_j, s_j : j = 1, \ldots, n\}$ is an orthonormal system in L^2, where

$$c_0(p) = \frac{1}{(2\pi)^{n/2}}, \quad c_j(p) = \frac{\sqrt{2}}{(2\pi)^{n/2}} \cos p_j, \quad s_j(p) = \frac{\sqrt{2}}{(2\pi)^{n/2}} \sin p_j, \quad j = 1, \ldots, n.$$

Adopting $V = V^{+}_{\lambda\mu} + V^{-}_{\lambda}$, we can see that the restriction $H^{+}_{\lambda\mu}$ (resp. H^{-}_{λ}) of the operator $H_{\lambda\mu}$ to L^2_{+} (resp. L^2_{-}) acts with the form

$$H^{+}_{\lambda\mu} = H_0 - V^{+}_{\lambda\mu} \quad (\text{resp. } H^{-}_{\lambda} = H_0 - V^{-}_{\lambda}).$$

Hence $H_{\lambda\mu}$ is decomposed as $H_{\lambda\mu} = H^{+}_{\lambda\mu} \oplus H^{-}_{\lambda}$ under $L^2 = L^2_{+} \oplus L^2_{-}$. We have $\sigma_{\mathrm{ess}}(H_{\lambda\mu}) = \sigma_{\mathrm{ac}}(H_{\lambda\mu}) = [0, 2n]$. Then we are interested in considering point spectrum of $H_{\lambda\mu}$ and studying their behaviors as two parameters λ and μ are varied.

2 Spectrum of even part $H^{+}_{\lambda\mu}$

The Birman-Schwinger principle helps us to reduce the problem to the study of spectrum of a finite dimensional linear operator. Denote by $(H_0 - z)^{-1}$ the resolvent of H_0, where $z \in \mathbb{C} \setminus [0, 2n]$. We can see that $(H_0 - z)^{-1} V^{+}_{\lambda\mu}$ is a finite rank operator. Let M_{n+1} denote the linear hull of $\{c_0, \cdots, c_n\}$. Then M_{n+1} is an $(n+1)$-dimensional subspace of L^2_{+}. We define $\tilde{M}_{n+1} = (H_0 - z)^{-1} M_{n+1}$ for $z \in \mathbb{C} \setminus [0, 2n]$. Then $\tilde{M}_{n+1}$ is also an $(n+1)$-dimensional subspace of L^2_{+} since $(H_0 - z)^{-1}$ is invertible. We define $C_1 : \mathbb{C}^{n+1} \to L^2_{+}$ by the map

$$C_1 : \mathbb{C}^{n+1} \ni (w_0, \cdots, w_n)^T \mapsto (H_0 - z)^{-1} \left(\mu w_0 c_0 + \frac{\lambda}{2} \sum_{j=1}^{n} w_j c_j \right) \in \tilde{M}_{n+1},$$

and define $C_2 : L^2_+ \to \mathbb{C}^{n+1}$ by

$$C_2 : L^2_+ \ni \phi \mapsto (\langle \phi, c_0 \rangle, \cdots, \langle \phi, c_n \rangle)^T \in \mathbb{C}^{n+1}.$$

Then we have the sequence of maps: $L^2_+ \xrightarrow{C_2} \mathbb{C}^{n+1} \xrightarrow{C_1} L^2_+$. In particular

$$(H_0 - z)^{-1} V^+_{\lambda\mu} = C_1 C_2.$$

Define $G_+(z) = C_2 C_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$.

Lemma 2.1 (The Birman-Schwinger principle for $z \in \mathbb{C} \setminus [0, 2n]$**)**

(1) $z \in \mathbb{C} \setminus [0, 2n]$ *is an eigenvalue of* $H^+_{\lambda\mu}$ *if and only if* $1 \in \sigma(G_+(z))$.

(2) If $z \in \mathbb{C} \setminus [0, 2n]$ *and* (λ, μ) *satisfy* $\det(G_+(z) - \mathrm{I}) = 0$. *Then* $Z = (w_0, \ldots, w_n)^T \in \mathbb{C}^{n+1}$ *is an eigenvector of* $G_+(z)$ *associated with eigenvalue* 1 *if and only if* $f = C_1 Z$, *i.e.*

$$f(p) = \frac{1}{(2\pi)^{n/2}} \frac{1}{E(p) - z} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{n} w_j \cos p_j \right) \tag{2.1}$$

is an eigenfunction of $H^+_{\lambda\mu}$ *associated with eigenvalue* z.

We consider the Birman-Schwinger principle for $z = 0$, which is the edge of the continuous spectrum of $H^+_{\lambda\mu}$, and it is the main issue to specify whether it is eigenvalue or threshold of $H^+_{\lambda\mu}$. In order to discuss $z = 0$ we extend the eigenvalue equation $H^+_{\lambda\mu} f = 0$ in L^2_+ to that in L^1_+. Note that $L^2_+ \subset L^1_+$. We consider the equation

$$E(p) f(p) - \frac{\mu}{(2\pi)^n} \int_{\mathbb{T}^n} f(p) dp - \frac{\lambda}{(2\pi)^n} \sum_{j=1}^{n} \cos p_j \int_{\mathbb{T}^n} \cos p_j f(p) dp = 0 \tag{2.2}$$

in the Banach space L^1_+. Conveniently, we describe (2.2) as $H^+_{\lambda\mu} f = 0$. Since we consider a solution $f \in L^1_+$, the integrals $\int_{\mathbb{T}^n} f(p) dp$ and $\int_{\mathbb{T}^n} \cos p_j f(p) dp$ are finite for $j = 1, \ldots, n$. The unique singular point of $1/E(p)$ is $p = 0$, and in the neighborhood of $p = 0$, we have $E(p) \approx |p|^2$. Then the following is fundamental, and its proof is straightforward.

Lemma 2.2 *Let* $h(p) = \varphi(p)/E(p)$, *where* $\varphi \in C(\mathbb{T}^n)$. *Then (1)-(5) follow.*

(1) It follows that $h \in L^2$ *for* $n \geq 5$, *and* $h \in L^1$ *for* $n \geq 3$.

(2) Let $1 \leq n \leq 4$ *and* $h \in L^2$. *Then* $\varphi(0) = 0$.

(3) Let $1 \leq n \leq 4$, $|\varphi(p)| < C|p|^{\alpha_n}$ *for some* $C > 0$ *and* $\alpha_n > \frac{4-n}{2}$. *Then* $h \in L^2$.

(4) Let $n = 1, 2$ *and* $h \in L^1$. *Then* $\varphi(0) = 0$.

(5) Let $n = 1, 2$, $|\varphi(p)| < C|p|^{\alpha_n}$ *for some* $C > 0$ *and* $\alpha_n > 2 - n$. *Then* $h \in L^1$.

Operator H_0^{-1} is not bounded in L^2_+ as well as in L^1_+. It is however obvious by Lemma 2.2 and $V^+_{\lambda\mu}f \in C(\mathbb{T}^n)$ that

$$L^2_+ \ni f \mapsto H_0^{-1}V^+_{\lambda\mu}f \in L^2_+, \quad n \geq 5, \tag{2.3}$$
$$L^1_+ \ni f \mapsto H_0^{-1}V^+_{\lambda\mu}f \in L^1_+, \quad n \geq 3. \tag{2.4}$$

Thus for $n \geq 3$ we can extend operators C_1 and C_2 . Let $n \geq 3$ and $Z = (w_0, \cdots, w_n)^T$. $\bar{C}_1 : \mathbb{C}^{n+1} \to L^1_+$ is defined by

$$\bar{C}_1 Z = \frac{1}{(2\pi)^{n/2}} \frac{1}{E(p)} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \cos p_j \right)$$

and $\bar{C}_2 : L^1_+ \to \mathbb{C}^{n+1}$ by $\bar{C}_2 : L^1_+ \ni \phi \mapsto (\int_{\mathbb{T}^n} \phi(p) c_0 dp, \cdots, \int_{T^n} \phi(p) c_n(p) dp)^T \in \mathbb{C}^{n+1}$. Then $\overline{C}_1\overline{C}_2 : L^1_+ \to L^1_+$. Consequently $G_+(0) = \overline{C}_2\overline{C}_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. Let $n \geq 3$. $\lim_{z\to 0} G_+(z) = G_+(0)$ and $\sigma(H_0^{-1}V^+_{\lambda\mu}) \setminus \{0\} = \sigma(G_+(0)) \setminus \{0\}$ follow.

Lemma 2.3 (Birman-Schwinger principle for $z = 0$**)** *Let $n \geq 3$. Then (1) and (2) follow.*

(1) Equation $H^+_{\lambda\mu} f = 0$ has a solution in L^1 if and only if $1 \in \sigma(G_+(0))$.

(2) Let $Z = (w_0, \ldots, w_n)^T \in \mathbb{C}^{n+1}$ be the solution of $G_+(0)Z = Z$ if and only if $f = \overline{C}_1 Z$, i.e.

$$f(p) = \frac{1}{(2\pi)^{n/2}} \frac{1}{E(p)} \left(\mu w_0 + \frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \cos p_j \right) \tag{2.5}$$

is a solution of $H^+_{\lambda\mu} f = 0$, where $w_0, \cdots, w_n$ are actually described by

$$w_0 = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{T}^n} f(p) dp, \quad w_j = \frac{\sqrt{2}}{(2\pi)^{n/2}} \int_{\mathbb{T}^n} f(p) \cos p_j dp, \quad j = 1, \ldots, n. \tag{2.6}$$

By the Birman-Schwinger principle in what follows we focus on investigating the spectrum of $G_+(z)$. Since $G_+(z)$ is defined for $z \in (-\infty, 0)$ for $n = 1, 2$, and $z \in (-\infty, 0]$ for $n \geq 3$. Hence in this section we suppose that $z \in \begin{cases} (-\infty, 0) & n = 1, 2, \\ (-\infty, 0] & n \geq 3. \end{cases}$ As the function $E(p) = E(p_1, \ldots, p_n)$ is invariant with respect to the permutations of its arguments $p_1, \ldots, p_n$, the integrals used for studying the spectrum of $G_+(z)$:

$$a(z) = \langle c_0, (H_0 - z)^{-1} c_0 \rangle, \quad b(z) = \frac{1}{\sqrt{2}} \langle c_0, (H_0 - z)^{-1} c_j \rangle \tag{2.7}$$
$$c(z) = \frac{1}{2} \langle c_j, (H_0 - z)^{-1} c_j \rangle, \quad d(z) = \frac{1}{2} \langle c_i, (H_0 - z)^{-1} c_j \rangle, \quad i \neq j, \tag{2.8}$$
$$s(z) = \frac{1}{2} \langle s_j, (H_0 - z)^{-1} s_j \rangle. \tag{2.9}$$

also do not depend on the particular choice of indices $0 \le i, j \le n$. Note that $a(z), b(z), c(z)$ and $s(z)$ are defined for $n \ge 1$ but $d(z)$ for $n \ge 2$. Hence the $(n+1) \times (n+1)$ matrix $G_+(z)$ has the form

$$G_+(z) = \begin{pmatrix} \mu a(z) & \frac{\lambda}{\sqrt{2}} b(z) & \dots & \dots & \frac{\lambda}{\sqrt{2}} b(z) \\ \sqrt{2}\mu b(z) & \lambda c(z) & \lambda d(z) & \dots & \lambda d(z) \\ \vdots & \lambda d(z) & \ddots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \lambda d(z) \\ \sqrt{2}\mu b(z) & \lambda d(z) & \dots & \lambda d(z) & \lambda c(z) \end{pmatrix}. \tag{2.10}$$

In order to study the eigenvalue 1 of $G_+(z)$ we calculate the determinant of $G_+(z) - \mathrm{I}$. We have $\det(G_+(z) - \mathrm{I}) = \delta_r(\lambda, \mu; z)\delta_c(\lambda; z)$, where

$$\delta_r(\lambda, \mu; z) = \begin{cases} \big(1 - \mu a(z)\big)\Big\{1 - \lambda\big(c(z) + (n-1)d(z)\big)\Big\} - n\lambda\mu b^2(z), & n \ge 2 \\ \big(1 - \mu a(z)\big)\big(1 - \lambda c(z)\big) - \lambda\mu b^2(z), & n = 1, \end{cases} \tag{2.11}$$

$$\delta_c(\lambda; z) = \begin{cases} \{\lambda(c(z) - d(z)) - 1\}^{n-1}, & n \ge 2 \\ 1, & n = 1. \end{cases} \tag{2.12}$$

We set

$$\alpha(z) = \begin{cases} c(z) + (n-1)d(z), & n \ge 2 \\ c(z), & n = 1, \end{cases} \qquad \gamma(z) = a(z)\alpha(z) - nb^2(z). \tag{2.13}$$

Functions $a(z)$, $\alpha(z)$, $\gamma(z)$, $b(z)$, $c(z) - d(z)$ and $s(z)$ are monotone increasing and positive in $(-\infty, 0]$. Moreover, their limits tend to zero as z tends to $-\infty$. Note that $c(z) - d(z)$ is considered only in the case of $n \ge 2$. The following relations hold:

$$\begin{aligned} a(z)s(z) &= b(z), & n &= 1, z < 0, \\ a(z)s(z) &< b(z), & n &= 2, z < 0, \\ a(z)s(z) &< b(z), & n &\ge 3, z \le 0, \\ c(z) - d(z) &< s(z), & n &\ge 2, z \le 0. \end{aligned} \tag{2.14}$$

The function $a(z)/b(z)$ is monotone decreasing in $(-\infty, 0]$, and there exist limits:

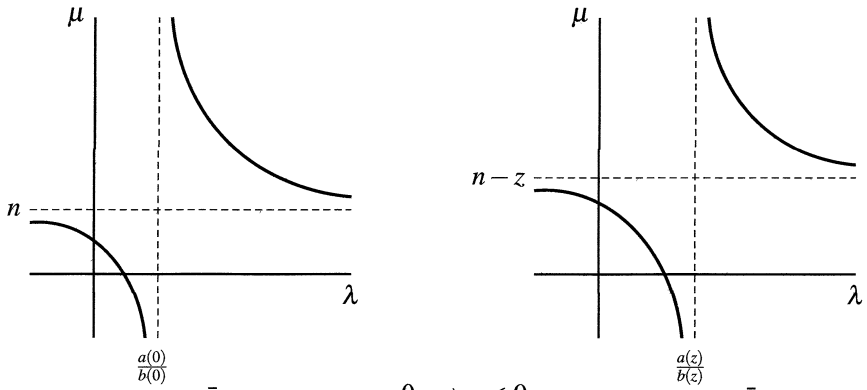
$$\lim_{z \to -\infty} \frac{a(z)}{b(z)} = +\infty, \quad \lim_{z \to 0-} \frac{a(z)}{b(z)} = \begin{cases} 1, & \text{for } n = 1, 2, \\ \frac{a(0)}{b(0)}, & \text{for } n \ge 3. \end{cases} \tag{2.15}$$

We extend $\delta_r(\lambda, \mu; \cdot)$ and $\delta_c(\lambda; \cdot)$, and discuss zeros of them to specify the eigenvalue of $H^+_{\lambda\mu}$. Let $z \in (-\infty, 0)$. Applying notation in (2.13), we describe $\delta_r(\lambda, \mu; z)$ as

$$\delta_r(\lambda, \mu; z) = \gamma(z)\mathbb{H}_z(\lambda, \mu) \tag{2.16}$$

where

$$\mathbb{H}_z(\lambda, \mu) = \Big(\lambda - \frac{a(z)}{b(z)}\Big)\Big(\mu - (n - z)\Big) - n. \tag{2.17}$$



Hyperbola $\bar{\mathbb{H}}_0$ $\quad z=0 \Longrightarrow z<0 \quad$ Hyperbola $\bar{\mathbb{H}}_z$

Figure 2: Hyperbola $\bar{\mathbb{H}}_z$

Instead of the equation $\delta_r(\lambda,\mu;z)=0$, relation (2.16) allows us to study the family of rectangular hyperbola $\mathbb{H}_z$ indexed by z,i.e. equilateral hyperbola $\mathbb{H}_z$ on (λ,μ)-plane, which is defined by

$$\mathbb{H}_z = \{(\lambda,\mu) \in \mathbb{R}^2 | \mathbb{H}_z(\lambda,\mu) = 0\}$$

with asymptote $(\lambda_\infty(z),\mu_\infty(z)) = (a(z)/b(z), n-z)$. $\mathbb{H}_z(\lambda,\mu)$ can be extended to $z \in (-\infty,0]$ for any dimension $n \geq 1$ as

$$\bar{\mathbb{H}}_z(\lambda,\mu) = \begin{cases} \mathbb{H}_z(\lambda,\mu), & z<0, \\ (\lambda - X)(\mu - n) - n, & z=0. \end{cases} \tag{2.18}$$

Here $X=1$ for $n=1,2$ and $X=a(0)/b(0)$ for $n \geq 3$. Refer to see Figure 2. Note that $\bar{\mathbb{H}}_z(0,0) = \frac{1}{b(z)} > 0$ for $z<0$. We also extend the family of hyperbola $\mathbb{H}_z$, $z \in (-\infty,0)$, to that of hyperbola $\bar{\mathbb{H}}_z$ indexed by $z \in (-\infty,0]$ by

$$\bar{\mathbb{H}}_z = \{(\lambda,\mu) \in \mathbb{R} \times \mathbb{R} | \bar{\mathbb{H}}_z(\lambda,\mu) = 0\}.$$

For any $z_1 < z_2$, $z_1,z_2 \in (-\infty,0]$, we note that the hyperbola $\bar{\mathbb{H}}_{z_1}$ can be moved to $\bar{\mathbb{H}}_{z_2}$ in parallel by the vector $g = \begin{pmatrix} \lambda_\infty(z_2) - \lambda_\infty(z_1) \\ \mu_\infty(z_2) - \mu_\infty(z_1) \end{pmatrix}$ whose components are positive. Refer to see Figure 3

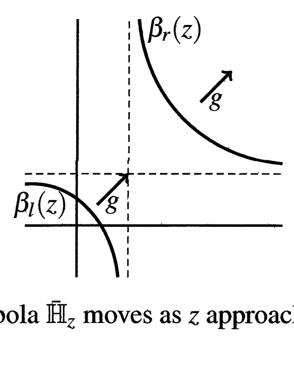


Figure 3: Hyperbola $\bar{\mathbb{H}}_z$ moves as z approaches to $-\infty$ from 0.

Let $\beta_l(z)$ (resp. $\beta_r(z)$) denote the left brunch (resp. the right brunch) of the hyperbola $\bar{\mathbb{H}}_z$, i.e. $\bar{\mathbb{H}}_z = \beta_l(z) \cup \beta_r(z)$, where $\cup$ denotes the disjoint union. We then see that for any $z_2 < z_1 \leq 0$ it follows that $\beta_l(z_1) \cap \beta_l(z_2) = \emptyset$ and $\beta_r(z_1) \cap \beta_r(z_2) = \emptyset$.

Lemma 2.4 *It follows that*

$$\begin{aligned}
(n=1)\quad \lim\nolimits_{z\to 0-} \delta_r(\lambda,\mu;z) &= \begin{cases} \infty & (\lambda,\mu) \notin \bar{\mathbb{H}}_0 \\ 1-\mu & (\lambda,\mu) \in \bar{\mathbb{H}}_0 \end{cases}, \\
(n=2)\quad \lim\nolimits_{z\to 0-} \delta_r(\lambda,\mu;z) &= \begin{cases} \infty & (\lambda,\mu) \notin \bar{\mathbb{H}}_0 \\ 1-\mu/2 & (\lambda,\mu) \in \bar{\mathbb{H}}_0 \end{cases}, \\
(n\geq 3)\quad \lim\nolimits_{z\to 0-} \delta_r(\lambda,\mu;z) &= b(0)\overline{\mathbb{H}}_0(\lambda,\mu).
\end{aligned}$$

Proof: In the case of $n \geq 3$ it is trivial to see that $\lim_{z\to 0-} \delta_r(\lambda,\mu;z) = b(0)\overline{\mathbb{H}}_0(\lambda,\mu)$. Then we consider cases of $n = 1,2$. We recall that

$$\delta_r(\lambda,\mu;z) = \gamma(z)\overline{\mathbb{H}}_z(\lambda,\mu) = \gamma(z)\overline{\mathbb{H}}_0(\lambda,\mu) + \gamma(z)(\overline{\mathbb{H}}_z(\lambda,\mu) - \overline{\mathbb{H}}_0(\lambda,\mu))$$

and

$$\gamma(z) = b(z) = a(z) - \frac{1}{n} - \frac{1}{n}za(z).$$

We can also directly see that for $n = 1,2$

$$\overline{\mathbb{H}}_z(\lambda,\mu) - \overline{\mathbb{H}}_0(\lambda,\mu) = -\frac{1}{nb(z)}(1+za(z))(\mu-n) + z(\lambda - \frac{a(z)}{b(z)}).$$

Together with them we have

$$\delta_r(\lambda,\mu;z) = (a(z) - \frac{1+za(z)}{n})\overline{\mathbb{H}}_0(\lambda,\mu) + \frac{a(z)}{b(z)}(1+za(z))(\frac{n-\mu}{n}) + \xi,$$

where

$$\xi = -za(z)(\lambda - \frac{a(z)}{b(z)}) + \frac{1+za(z)}{n}\left(\frac{(1+za(z))(\mu-n)}{nb(z)} + z(\lambda - \frac{a(z)}{b(z)})\right).$$

It is well known in [5] that

$$\begin{aligned}
a(z) &= \frac{1}{\sqrt{-z}\sqrt{2-z}}, \quad n=1, \\
a(z) &= -\frac{\sqrt{2}}{2\pi}\ln(-z) + (\frac{1}{2} - \frac{\sqrt{2}}{\pi}) + O(-z), \quad \text{as} \quad z\to 0-, \quad n=2.
\end{aligned}$$

By this it is crucial to see that

$$\lim_{z\to 0-} za(z) = 0, \quad \lim_{z\to 0-} b(z) = \infty, \quad \lim_{z\to 0-} \frac{a(z)}{b(z)} = 1$$

and

$$\lim_{z\to 0-} \xi = 0, \quad \lim_{z\to 0-} \frac{a(z)}{b(z)}(1+za(z))(\frac{n-\mu}{n}) = 1 - \frac{\mu}{n}$$

for $n=1,2$. Let $n=1$. Then

$$\delta_r(\lambda,\mu;z)=(a(z)-1-za(z))\bar{\mathbb{H}}_0(\lambda,\mu)+\frac{a(z)}{b(z)}(1+za(z))(1-\mu)+\xi$$

and the corollary follows for $n=1$. Let $n=2$. In a similar manner to the case of $n=1$ we have

$$\delta_r(\lambda,\mu;z)=(a(z)-\frac{1+za(z)}{2})\bar{\mathbb{H}}_0(\lambda,\mu)+\frac{a(z)}{b(z)}(1+za(z))(1-\frac{\mu}{2})+\xi,$$

and the corollary follows for $n=2$. Hence the proof of the corollary can be derived.

We define $\bar{\delta}_r(\lambda,\mu;z)$ for $z\in(-\infty,0]$ by

$$\bar{\delta}_r(\lambda,\mu;z)=\begin{cases}\delta_r(\lambda,\mu;z), & z\in(-\infty,0),\\ \lim_{z\to 0-}\delta_r(\lambda,\mu;z), & z=0.\end{cases}\tag{2.19}$$

From Lemma 2.4 we can see that $\bar{\delta}_r(\lambda,\mu;z)$ converges to

$$\bar{\delta}_r(\lambda,\mu;0)=\begin{cases}1-\mu, & n=1,\ (\lambda,\mu)\in\bar{\mathbb{H}}_0,\\ 1-\mu/2, & n=2,\ (\lambda,\mu)\in\bar{\mathbb{H}}_0,\\ 0, & n\geq 3,\ (\lambda,\mu)\in\bar{\mathbb{H}}_0.\end{cases}\tag{2.20}$$

Then we can show the continuity of $\bar{\delta}_r(\lambda,\mu;z)$ on z:

Lemma 2.5 *It follows that*

($n=1,2$) $\bar{\delta}_r(\lambda,\mu;z)$ is continuous in $z\in(-\infty,0]$ for $(\lambda,\mu)\in\bar{\mathbb{H}}_0$,

($n\geq 3$) $\bar{\delta}_r(\lambda,\mu;z)$ is continuous in $z\in(-\infty,0]$ for $(\lambda,\mu)\in\mathbb{R}^2$.

We set $\beta_l=\beta_l(0)$ and $\beta_r=\beta_r(0)$. The brunches β_l and β_r of the hyperbola $\bar{\mathbb{H}}_0$ split $\mathbb{R}^2$ into three open sets

$$\begin{aligned}G_0&=\{(\lambda,\mu)\in\mathbb{R}^2|\bar{\mathbb{H}}_0(\lambda,\mu)>0,\lambda<\lambda_\infty(0)\},\\ G_1&=\{(\lambda,\mu)\in\mathbb{R}^2|\bar{\mathbb{H}}_0(\lambda,\mu)<0\},\\ G_2&=\{(\lambda,\mu)\in\mathbb{R}^2|\bar{\mathbb{H}}_0(\lambda,\mu)>0,\lambda>\lambda_\infty(0)\}.\end{aligned}$$

Refer to see Figure 4. Hence $\partial G_0=\beta_l$ and $\partial G_2=\beta_r$ follow.

Lemma 2.6 *(1)*
1. *Let $(\lambda,\mu)\in G_0\cup\beta_l$. Then $\bar{\delta}_r(\lambda,\mu;z)\neq 0$ for $z\in(-\infty,0)$.*
2. *Let $(\lambda,\mu)\in\beta_l$. Then $\bar{\delta}_r(\lambda,\mu;0)\neq 0$ for $n=1,2$.*
3. *Let $(\lambda,\mu)\in\beta_l$. Then $\bar{\delta}_r(\lambda,\mu;0)=0$ for $n\geq 3$.*

(2)
1. *Let $(\lambda,\mu)\in G_1\cup\beta_r$. Then $\exists_1 z\in(-\infty,0)$ such that $\bar{\delta}_r(\lambda,\mu;z)=0$.*
2. *Let $(\lambda,\mu)\in\beta_r$. Then $\bar{\delta}_r(\lambda,\mu;0)\neq 0$ for $n=1,2$.*
3. *Let $(\lambda,\mu)\in\beta_r$. Then $\bar{\delta}_r(\lambda,\mu;0)=0$ for $n\geq 3$.*

(3) Let $(\lambda,\mu)\in G_2$. Then $\exists z_1,z_2\in(-\infty,0)$ such that $\bar{\delta}_r(\lambda,\mu;z_1)=\bar{\delta}_r(\lambda,\mu;z_2)=0$.

Proof: Let $(\lambda,\mu)\in G_0\cup\beta_l$. Then we can see that $(\lambda,\mu)\notin\bar{\mathbb{H}}_z$ for any $z\in(-\infty,0)$. Thus (1)1 follows. Also it can be seen that $\cup_{z\in(-\infty,0)}\beta_l(z)\supset G_1$, $\beta_l(z)\cap\beta_l(w)=\emptyset$ if $z\neq w$, and $\beta_r(z)\cap G_1=\emptyset$. Hence there exists a unique $z\in(-\infty,0)$ such that $(\lambda,\mu)\in\beta_l(z)$, which proves (2)1. We can also see that $\cup_{z\in(-\infty,0)}\beta_l(z)\supset G_2$, $\cup_{z\in(-\infty,0)}\beta_r(z)\supset G_2$, $\beta_l(z)\cap\beta_l(w)=\emptyset$ if $z\neq w$, $\beta_r(z)\cap\beta_r(w)=\emptyset$ if $z\neq w$, and $\beta_l(z)\cap\beta_r(z)=\emptyset$. Hence there exist $z_1,z_2\in(-\infty,0)$ such that $(\lambda,\mu)\in\beta_l(z_1)$ and $(\lambda,\mu)\in\beta_r(z_2)$, which proves (3). Finally we note that since $(\lambda,\mu)\in\beta_l$ implies that $1\neq\mu$ for $n=1$, and $2\neq\mu$ for $n=2$, $\bar{\delta}_r(\lambda,\mu;0)\neq 0$ for $n=1,2$ follows. Hence (1) 2 and (1) 3 follow from (2.20), and (2) 2 and (2) 3 are similarly proven. **qed**

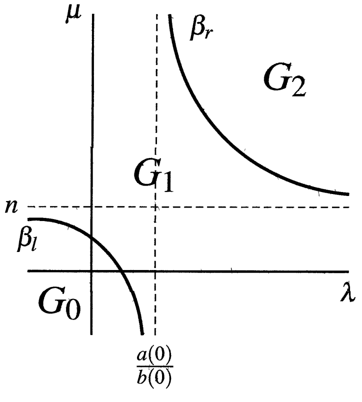


Figure 4: Region of G_j

By virtue of Lemma 2.6, $\bar{\delta}_r(\lambda,\mu;\cdot)$ has at most two zeros in $(-\infty,0)$.

Lemma 2.7 *Let $n\geq 1$. Then (1) and (2) follow.*

(1) Let $\lambda\neq 0$. We assume that $z_1,z_2\in(-\infty,0)$ and $\delta_r(\lambda,\mu;z_k)=0$ (if they exist). Then $1-\mu a(z_k)\neq 0$ and $G_+(z_k)Z_k=Z_k$ has the solutions: $Z_k=(\frac{\lambda}{\sqrt{2}}\frac{nb(z_k)}{1-\mu a(z_k)},1,\cdots,1)$, $k=1,2$, and the corresponding eigenfunctions, $H^+_{\lambda\mu}f_k=zf_k$, are

$$f_k(p)=\frac{\lambda}{\sqrt{2}}\frac{1}{(2\pi)^{n/2}}\frac{1}{E(p)-z_k}\Big(\mu\frac{nb(z_k)}{1-\mu a(z_k)}+\sum_{j=1}^{n}\cos p_j\Big),\quad k=1,2. \tag{2.21}$$

(2) Let $\lambda=0$. We assume that $z\in(-\infty,0)$ and $\delta_r(0,\mu;z)=0$. Then $1-\mu a(z)=0$ and $G_+(z)Z=Z$ has the solution: $Z=(1,\sqrt{2}\mu b(z),\cdots,\sqrt{2}\mu b(z))^T$ and the corresponding eigenfunction, $H^+_{\lambda\mu}f=zf$, is

$$f(p)=\frac{\mu}{(2\pi)^{n/2}}\frac{1}{E(p)-z} \tag{2.22}$$

Proof: We prove the case of $n\geq 2$. The proof for the case of $n=1$ is similar. Since $\delta_r(\lambda,\mu;z)=0$, we see that

$$(1-\mu a(z))\Big(1-\lambda\big(c(z)+(n-1)d(z)\big)\Big)-n\lambda\mu b^2(z)=0.$$

Then $1-\mu a(z)\neq 0$ if and only if $\lambda\neq 0$, and we also have the algebraic relation

$$1-\lambda\big(c(z)+(n-1)d(z)\big)=\frac{n\lambda\mu b^2(z)}{1-\mu a(z)}.$$

From this relation it follows that $G_+(z_k)Z_k=Z_k$ for $\lambda\neq 0$. In the case of $\lambda=0$ we can prove the lemma in a similar way. **qed**

We study zeros of $\delta_c(\lambda;z)$. In a similar manner we extend $\delta_c(\lambda,z)$ for $z\in(-\infty,0]$. When $n\geq 2$, the function $c(z)-d(z)$ exists, and we can define $\alpha=\lim_{z\to 0-}c(z)-d(z)$. Note that $\alpha>0$ and we set $\lambda_c=\frac{1}{\alpha}$. Let us write $\delta_c(\lambda;z)=\rho(\lambda;z)^{n-1}$, where $\rho(\lambda;z)=\lambda(c(z)-d(z))-1$. We define $\bar{\delta}_c(\lambda;z)$ by

$$\bar{\delta}_c(\lambda;z)=\begin{cases}\delta_c(\lambda;z), & z\in(-\infty,0), n\geq 1,\\ (\lambda\alpha-1)^{n-1}, & z=0, \quad n\geq 2,\\ 1, & z=0, \quad n=1.\end{cases}$$

Lemma 2.8 *Let $n\geq 2$. Then (1)–(3) follow. (1) Let $\lambda\leq\lambda_c$. Then $\rho(\lambda;z)\neq 0$ for any $z\in(-\infty,0)$. (2) Let $\lambda=\lambda_c$. Then $\rho(\lambda;0)=0$. (3) Let $\lambda>\lambda_c$. Then there exists unique $z\in(-\infty,0)$ such that $\rho(\lambda;z)=0$ with multiplicity one.*

Proof: Since $c(z)-d(z)>0$ is strictly monotone increasing in $(-\infty,0)$, we get

$$\rho(\lambda;z)\leq\rho(\lambda_c;z)<\rho(\lambda_c;0)=0,\quad \text{if}\quad 0<\lambda\leq\lambda_c,$$
$$\rho(\lambda;z)=-1,\quad \text{if}\quad \lambda=0,$$

which prove (1) and (2). Since $\rho(\lambda;0)>\rho(\lambda_c;0)=0$ and $\lim_{z\to-\infty}\rho(\lambda;z)=-1$ there exists $z\in(-\infty,0)$ such that $\rho(\lambda;z)=0$. By the monotonicity of $\rho(\lambda;\cdot)$ this zero is a unique and has multiplicity one. Hence (3) is proven. **qed**

We immediately have a lemma.

Lemma 2.9 *Let $n\geq 2$. Then (1)–(3) follow.*

(1) For any $\lambda\leq\lambda_c$, $\bar{\delta}_c(\lambda;\cdot)$ has no zero in $(-\infty,0)$.

(2) Let $\lambda=\lambda_c$. Then $\bar{\delta}_c(\lambda;0)=0$, and $z=0$ has multiplicity $n-1$.

(3) For any $\lambda>\lambda_c$, $\bar{\delta}_c(\lambda;\cdot)$ has a unique zero in $(-\infty,0)$ with multiplicity $n-1$.

Next we show the eigenfunction corresponding to zeros of $\delta_c(\lambda;\cdot)$.

Lemma 2.10 *Let $n\geq 2$, $z\in(-\infty,0)$ and $\bar{\delta}_c(\lambda;z)=0$. I.e., $\lambda=\frac{1}{c(z)-d(z)}$. Then the solutions of $G_+(z)Z=Z$ are given by $Z_j=(0,1,0,\cdots,\overset{j+2}{-1},\cdots,0)^T$, $j=1,...,n-1$, and hence the corresponding eigenfunctions, $H^+_{\lambda\mu}g_j=zg_j$, are*

$$g_j(p)=\frac{\lambda}{\sqrt{2}}\frac{1}{(2\pi)^{n/2}}\frac{1}{E(p)-z}(\cos p_1-\cos p_{j+1}),\quad j=1,\dots,n-1. \tag{2.23}$$

In particular the multiplicity of eigenvalue z is at least $n-1$.

Now we study the spectrum located on the left edge of the essential spectrum $[0,2n]$, i.e., $z=0$. Suppose that $(\lambda,\mu)\in\bar{\mathbb{H}}_0$. Then it is possibly $\bar{\delta}_r(\lambda,\mu;0)=0$ or $\bar{\delta}_c(\lambda,\mu;0)=0$. We study zeros of $\bar{\delta}_r(\lambda,\mu;0)$ for $n\geq 3$. We set $a(0)=a$ and $b(0)=b$, and both a and b are finite for $n\geq 3$.

Lemma 2.11 *Let $n\geq 3$. (1) Let $\lambda\neq 0$ and $\bar{\delta}_r(\lambda,\mu;0)=0$. Then $1-\mu a\neq 0$ and $G_+(0)Z=Z$ has the solution $Z=(\frac{\lambda}{\sqrt{2}}\frac{nb}{1-\mu a},1,\cdots,1)^T$ and the corresponding equation $H^+_{\lambda\mu}f=0$ has the solution:*

$$f(p)=\frac{\lambda}{\sqrt{2}}\frac{1}{(2\pi)^{n/2}}\frac{1}{E(p)}\Big(\mu\frac{nb}{1-\mu a}+\sum_{j=1}^{n}\cos p_j\Big). \tag{2.24}$$

(2) Let $\lambda=0$ and $\delta_r(0,\mu;0)=0$. Then $1-\mu a=0$ and $G_+(0)Z=Z$ has the solution: $Z=(1,0,\cdots,0)^T$ and the corresponding equation $H^+_{\lambda\mu}f=0$ has the solution:

$$f(p)=\frac{\mu}{(2\pi)^{n/2}}\frac{1}{E(p)}. \tag{2.25}$$

Proof: The proof is the same as that of Lemma 2.7. **qed**

Next we show the solution corresponding to zeros of $\delta_c(\lambda;\cdot)$. Similar to the case of $\delta_r(\lambda,\mu;z)=0$, we have the lemma below.

Lemma 2.12 *Let $n\geq 2$ and $\delta_c(\lambda;0)=0$, i.e., $\lambda=\lambda_c$. Then the solutions of $G_+(0)Z=Z$ are given by $Z_j=(0,1,0,\cdots,\overset{j+2}{-1},\cdots,0)^T$, $j=1,...,n-1$, and hence the corresponding equation $H^+_{\lambda\mu}g_j=0$ has the solutions*

$$g_j(p)=\frac{\lambda_c}{\sqrt{2}}\frac{1}{(2\pi)^{n/2}}\frac{1}{E(p)}(\cos p_1-\cos p_{j+1}),\quad j=1,\ldots,n-1. \tag{2.26}$$

Proof: The proof is the same as that of Lemma 2.10. **qed**

As was seen above the problem for $n\geq 3$ can be reduced to study the spectrum of G_+ by the Birman-Schwinger principle, the problem for $n=1,2$ should be however directly investigated.

As was seen above the problem for $n\geq 3$ can be reduced to study the spectrum of G_{e} by the Birman-Schwinger principle, the problem for $n=1,2$ should be however directly investigated.

Let f be a solution of $H^+_{\lambda\mu}f=0$ (resp. $H^-_{\lambda}f=0$) (1) If $f\in L^2_+$ (resp. $f\in L^2_-$), we say that 0 is a threshold eigenvalue of $H^+_{\lambda\mu}$ (resp. H^-_{λ}). (2) If $f\in L^1_+\setminus L^2_+$ (resp. $f\in L^1_-\setminus L^2_-$), we say that 0 is a threshold resonance of $H^+_{\lambda\mu}$ (resp. H^-_{λ}). (3) If $f\in L^{\varepsilon}_+\setminus L^1_+$ (resp. $f\in L^{\varepsilon}_-\setminus L^1_-$) for any $0<\varepsilon<1$, we say that 0 is a super-threshold resonance of $H^+_{\lambda\mu}$ (resp. H^-_{λ}).

Lemma 2.13 *Let $n=1$.*

(1) Suppose that $f\in L^1(\mathbb{T})$ and $H^{\mathrm{e}}_{\lambda\mu}f=0$. Then $f=0$. In particular $H^{\mathrm{e}}_{\lambda\mu}$ has no threshold resonance.

(2) There is no non-zero f such that $f \in L^{\varepsilon}(\mathbb{T}^2)\setminus L^1(\mathbb{T}^2)$ for some $0<\varepsilon<1$ and $H^{\mathrm{e}}_{\lambda\mu}f=0$. In particular $H^{\mathrm{e}}_{\lambda\mu}$ has no super-threshold resonance.

Proof: (1) $H^{\mathrm{e}}_{\lambda\mu}f=0$ gives $f=\varphi/E$ and $\varphi(p)=\mu u_0+\lambda u_1\cos p$ by (2.2). From $f\in L^1(\mathbb{T})$ it follows that $\varphi(0)=\mu u_0+\lambda u_1=0$. Hence

$$f(p)=\frac{1}{E(p)}(1-\cos p)\mu u_0=\mu u_0.$$

We get $u_1=\mu u_0\frac{1}{2\pi}\int_{\mathbb{T}}\cos t dt=0$, which gives $\mu u_0=0$ and $f=0$.
(2) Since $f\notin L^1(\mathbb{T})$. It must be that $\mu=0$ and $f=\varphi/E$ with $\varphi(p)=\lambda u_1\cos p$. Hence

$$u_1=\frac{\lambda}{(2\pi)^2}\int_{\mathbb{T}}\frac{u_1\cos^2 p}{E(p)}dp.$$

Then $u_1=0$, since $\int_{\mathbb{T}}\frac{\cos^2 p}{E(p)}dp=\infty$. Then $f=0$ follows.

Next we discuss the spectrum of $H^{\mathrm{e}}_{\lambda\mu}$ for $n=2$ at the threshold

Lemma 2.14 *Let $n=2$.*

(1) Suppose that $f\in L^1(\mathbb{T}^2)$ and $H^{\mathrm{e}}_{\lambda\mu}f=0$. Then $\lambda=\lambda_c$ and

$$f(p)=\lambda_c u_1\frac{\cos p_1-\cos p_2}{E(p)}. \tag{2.27}$$

In particular $f\in L^2(\mathbb{T}^2)$ and $H^{\mathrm{e}}_{\lambda\mu}$ has no threshold resonance.

(2) There is no non-zero f such that $f\in L^{\varepsilon}(\mathbb{T}^2)\setminus L^1(\mathbb{T}^2)$ for some $0<\varepsilon<1$ and $H^{\mathrm{e}}_{\lambda\mu}f=0$. In particular $H^{\mathrm{e}}_{\lambda\mu}$ has no super-threshold resonance.

Proof: (1) Consider $H^{\mathrm{e}}_{\lambda\mu}f=0$ in $L^1(\mathbb{T}^2)$. We can take $f=\varphi/E$ and $\varphi(p)=\mu u_0+\lambda u_1\cos p_1+\lambda u_2\cos p_2$. Since $f\in L^1(\mathbb{T}^2)$, we get $\varphi(0)=\mu u_0+\lambda(u_1+u_2)=0$ and so

$$f(p)=\frac{\lambda}{E(p)}(-u_1(1-\cos p_1)-u_2(1-\cos p_2))$$

We obtain

$$u_1=-\frac{\lambda}{(2\pi)^2}\left(u_1\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_1)}{E(p)}dp+u_2\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_2)}{E(p)}dp\right),$$
$$u_2=-\frac{\lambda}{(2\pi)^2}\left(u_1\int_{\mathbb{T}^2}\frac{\cos p_2(1-\cos p_1)}{E(p)}dp+u_2\int_{\mathbb{T}^2}\frac{\cos p_2(1-\cos p_2)}{E(p)}dp\right).$$

Since $\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_1)}{E(p)}dp=-\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_2)}{E(p)}dp$, we get

$$u_1=\frac{\lambda}{(2\pi)^2}(u_2-u_1)\left(\int_{\mathbb{T}^2}\frac{\cos p_1(1-\cos p_1)}{E(p)}dp\right),$$
$$u_2=\frac{\lambda}{(2\pi)^2}(u_1-u_2)\left(\int_{\mathbb{T}^2}\frac{\cos p_2(1-\cos p_2)}{E(p)}dp\right)$$

and hence $u_1 = -u_2$. Consequently, $\mu u_0 = 0$, and the solution of $H^{\mathrm{e}}_{\lambda\mu} f = 0$ is of the form

$$f(p) = \lambda u_1 \frac{\cos p_1 - \cos p_2}{E(p)} \in L^2(\mathbb{T}^2). \tag{2.28}$$

Inserting this into the definition of u_1, we have $\frac{\lambda}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{\cos p_1(\cos p_1 - \cos p_2)}{E(p)} dp = 1$ and thus taking $\lambda = \lambda_c$ we can see that (2.27) is the solution of $H^{\mathrm{e}}_{\lambda\mu} f = 0$. Notice that $u_0 = 0$ follows from (2.28).
(2) Since $f \notin L^1(\mathbb{T}^2)$. It must be that $\mu = 0$ and $f = \varphi/E$ with $\varphi(p) = \lambda u_1 \cos p_1 + \lambda u_2 \cos p_2$. Hence

$$u_1 = \frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{u_1 \cos^2 p_1 + u_2 \cos p_1 \cos p_2}{E(p)} dp,$$
$$u_2 = -\frac{\lambda}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{u_1 \cos p_2 \cos p_1 + u_2 \cos^2 p_2}{E(p)} dp.$$

Then $u_1 = -u_2$ and $1 = \frac{\lambda}{(2\pi)^2}\int_{\mathbb{T}^2} \frac{\cos p_1(\cos p_1 + \cos p_2)}{E(p)} dp$. Thus $\lambda = \lambda_c$. Then f is given by (2.28), but $f \in L^2(\mathbb{T}^2)$. This contradicts with $f \notin L^1(\mathbb{T}^2)$.

Lemma 2.15 *(1)-(5) follow:*

(1) Let $n = 1$. Then 0 is none of a threshold eigenvalue, a threshold resonance and a super-threshold resonance.

(2) Let $n = 2$. Then 0 is a threshold eigenvalue with (2.26) for $(\lambda, \mu) = (\lambda_c, \mu)$ and its multiplicity is one.

(3) Let $n = 3, 4$. Suppose $(\lambda, \mu) \in \mathbb{H}_0$. Then 0 is a threshold resonance with eigenvector (2.24) for $\lambda \neq 0$, and (2.25) for $\lambda = 0$, i.e., $(\lambda, \mu) = (0, 1/a)$.

(4) Let $n = 3, 4$. Suppose $(\lambda, \mu) \in \mathbb{H}_0$. Then 0 is a threshold eigenvalue with (2.26) for $\lambda = \lambda_c$ and its with multiplicity is $n-1$.

(5) Let $n \geq 5$. Suppose $(\lambda, \mu) \in \mathbb{H}_0$. Then 0 is a threshold eigenvalue with eigenvector (2.24) for $\lambda_c \neq \lambda \neq 0$ and multiplicity one, (2.24) and (2.26) for $\lambda = \lambda_c$ and multiplicity n, and (2.25) for $\lambda = 0$, i.e., $(\lambda, \mu) = (0, 1/a)$, and multiplicity one.

Proof: (1) follows from Lemma 2.13. The solution of $H^{+}_{\lambda\mu} f = 0$ is given by (2.24), (2.25) and (2.26). We note that $\int_{|p|<\varepsilon} \frac{1}{E^2(p)} dp = \infty$ for $n = 2, 3, 4$ for any $\varepsilon > 0$, and $\int_{|p|<\varepsilon} \frac{1}{E^2(p)} dp < \infty$ for $n \geq 5$ for any $\varepsilon > 0$. Since $(\lambda, \mu) \in \bar{\mathbb{H}}_0$, $n \neq \mu$, and we can see that

$$\frac{nb\mu}{1-\mu a} + \sum_{j=1}^{n} \cos 0 = \frac{nb\mu}{1-\mu a} + n = n\left(\frac{1-\mu(a-b)}{1-\mu a}\right) = \frac{n-\mu}{1-\mu a} \neq 0.$$

Hence, using Lemma 2.2, we obtain (2.24), (2.25) $\in L^2$ for $n \geq 5$, (2.24), (2.25) $\in L^1 \setminus L^2$ for $n = 3, 4$ and (2.26) $\in L^2$ for $n \geq 2$. (2) follows from Lemmas 2.14 and 2.12. (3) and (5) follow from Lemmas 2.12 and 2.11. (4) follows from Lemma 2.12. **qed**

3 Spectrum of odd part H_λ^-

In the previous sections, we study the spectrum of $H_{\lambda\mu}^+$ by using the Birman-Schwinger principle for $n \geq 3$, and by directly solving $H_{\lambda\mu}^+ f = 0$ for $n = 1,2$. In the case of H_λ^- we can proceed in a similar way to the the case of $H_{\lambda\mu}^+$ and rather easier than that of $H_{\lambda\mu}^+$ as is seen below. Let $z \in \mathbb{C} \setminus [0,2n]$. Let N_n be the linear hull of $\{s_1, \cdots, s_n\}$. As is done for $H_{\lambda\mu}^+$, we can see that $(H_0 - z)^{-1} V_\lambda^- = S_1 S_2$. Here S_1 and S_2 are defined by

$$S_1 : \mathbb{C}^n \ni (w_1, \cdots, w_n) \mapsto (H_0 - z)^{-1} \frac{\lambda}{2} \sum_{j=1}^n w_j s_j \in L_-^2,$$
$$S_2 : L_-^2 \ni \phi \mapsto (\langle \phi, s_1 \rangle, \cdots, \langle \phi, s_n \rangle) \in \mathbb{C}^n.$$

We set $G_-(z) = S_2 S_1 : \mathbb{C}^n \to \mathbb{C}^n$. The following assertion is proved as Lemma 2.1.

Lemma 3.1 (Birman-Schwinger principle for $z \in \mathbb{C} \setminus [0,2n]$**)**

(1) $z \in \mathbb{C} \setminus [0,2n]$ *is an eigenvalue of* H_λ^- *if and only if* $1 \in \sigma(G_-(z))$.

(2) Let $z \in \mathbb{C} \setminus [0,2n]$ *and* $Z = (w_1, \ldots, w_n)^T \in \mathbb{C}^n$ *be such that* $G_-(z) Z = Z$. *Then* $f = S_1 Z$,

$$f(p) = \frac{1}{(2\pi)^{n/2}} \frac{1}{E(p) - z} \left(\frac{\lambda}{\sqrt{2}} \sum_{j=1}^n w_j \sin p_j \right)$$

is an eigenfunction of H_λ^-, *i.e.,* $H_\lambda^- f = zf$.

We see that $G_-(z) = \lambda s(z) I$. Consequently we have for $n \geq 1$, $\delta_s(\lambda; z) = \det(G_-(z) - I) = (\lambda s(z) - 1)^n$. Since $G_-(z)$ is diagonal, it is very easy to find solution of $G_-(z) Z = Z$. It has n independent solutions: $Z_j = (0, \cdots, \overset{j}{1}, \cdots 0)^T$. The corresponding eigenvector, $H_\lambda^- f_j = z f_j$, is given by

$$f_j(p) = \frac{1}{E(p) - z} \frac{1}{(2\pi)^{n/2}} \frac{\lambda}{\sqrt{2}} \sin p_j, \quad j = 1, \ldots, n, \tag{3.1}$$

where $\lambda = 1/s(z)$. In particular the multiplicity of z is n. We can extend the Birman-Schwinger principle for $z = 0$. We extend the eigenvalue equation $H_\lambda^- f = 0$ in L_-^2 to that in L_-^1. We consider the equation

$$E(p) f(p) - \frac{\lambda}{(2\pi)^n} \sum_{j=1}^n \sin p_j \int_{\mathbb{T}^n} \sin p_j f(p) dp = 0 \tag{3.2}$$

in L_-^1. We also describe (3.2) as $H_\lambda^- f = 0$. We can see that $\sin p_j / E(p) \approx 1/|p|$ in the neighborhood of $p = 0$, and then $\sin p_j / E(p) \in L^1$ for $n \geq 2$. By (5) of Lemma 2.2 and $V_\lambda^- f \in C$ we can see that

$$L_-^2 \ni f \mapsto H_0^{-1} V_\lambda^+ f \in L_-^2, \quad n \geq 3, \tag{3.3}$$
$$L_-^1 \ni f \mapsto H_0^{-1} V_\lambda^- f \in L_-^1, \quad n \geq 2. \tag{3.4}$$

Thus for $n \geq 2$ we can extend operators S_1 and S_2. Let $n \geq 2$ and $Z = (w_1, \cdots, w_n)^T$. $\bar{S}_1 : \mathbb{C}^n \to L^1_-$ is defined by

$$\bar{S}_1 Z = \frac{1}{(2\pi)^{n/2}} \frac{\lambda}{\sqrt{2}} \frac{1}{E(p)} \sum_{j=1}^{n} w_j \sin p_j$$

and $\bar{S}_2 : L^1_- \to \mathbb{C}^n$ by

$$\bar{S}_2 : L^1_- \ni \phi \mapsto \left(\int_{T^n} \phi(p) s_1(p) dp, \ldots, \int_{T^n} \phi(p) s_n(p) dp \right)^T \in \mathbb{C}^n.$$

Then $\overline{S}_1 \overline{S}_2 : L^1_- \to L^1_-$. Consequently $G_-(0) = \overline{S}_2 \overline{S}_1 : \mathbb{C}^n \to \mathbb{C}^n$ is described as an $n \times n$ matrix. Let $n \geq 2$. We have (1) $\lim_{z \to 0} G_-(z) = G_-(0)$, and (2) $\sigma(H_0^{-1} V_\lambda^-) \setminus \{0\} = \sigma(G_-(0)) \setminus \{0\}$. Hence for $n \geq 2$,

$$G_-(0) = \lambda s(0) I \tag{3.5}$$

and $\bar{\delta}_s(\lambda; z)$ is defined by

$$\bar{\delta}_s(\lambda; z) = \begin{cases} \delta_s(\lambda; z) & z \in (-\infty, 0), \\ (\lambda s(0) - 1)^n & z = 0. \end{cases} \tag{3.6}$$

Lemma 3.2 (Birman-Schwinger principle for $z = 0$**)** *Let $n \geq 2$. Then (1) and (2) follow. (1) Equation $H_\lambda^- f = 0$ has a solution in L^1 if and only if $1 \in \sigma(G_-(0))$. (2) Let $Z = (w_1, \ldots, w_n)^T \in \mathbb{C}^n$ be the solution of $G_-(0) Z = Z$ if and only if $f = \bar{S}_1 Z$, i.e.,*

$$f(p) = \frac{1}{(2\pi)^n} \frac{1}{E(p)} \frac{\lambda}{\sqrt{2}} \sum_{j=1}^{n} w_j \sin p_j$$

is a solution of $H_\lambda^- f = 0$, where $w_j = \frac{\sqrt{2}}{(2\pi)^{n/2}} \int_{\mathbb{T}^n} f(p) \sin p_j dp$, $j = 1, \ldots, n$.

Proof: The proof is the same as that of Lemma 2.3. **qed**

Set $\lambda_s = \frac{1}{s(0)}$.

Lemma 3.3 *Let $n \geq 1$. Then (1)-(3) follow:*

(1) Let $\lambda \leq \lambda_s$. Then $\bar{\delta}_s(\lambda; \cdot)$ has no zero in $(-\infty, 0)$.

(2) Let $\lambda = \lambda_s$. Then $\bar{\delta}_s(\lambda_s; 0) = 0$ and $z = 0$ has multiplicity n.

(3) Let $\lambda > \lambda_s$. Then $\bar{\delta}_s(\lambda; \cdot)$ has a unique zero in $(-\infty, 0)$ with multiplicity n.

Proof: The proof is similar to that of Lemma 2.9, and hence we omit it. **qed**

Threshold resonances and threshold eigenvalues for H_λ^- can be discussed by the Birman-Schwinger principle for $n \geq 2$.

Lemma 3.4 *Let $n \geq 2$. Then the solutions of equation $H_\lambda^- f = 0$ are given by*

$$f_j(p) = \frac{1}{(2\pi)^{n/2}} \frac{\lambda_s}{\sqrt{2}} \frac{\sin p_j}{E(p)}, \quad j = 1, \ldots, n. \tag{3.7}$$

Proof: By $\bar{\delta}(\lambda_s, 0) = 0$ and Lemma 3.2 the lemma follows. **qed**

For $n = 1$ we can directly see that $H_\lambda^- f = 0$ has no solution in L^1, but H_λ^- has a super-threshold resonance.

Proposition 3.5 (Super-threshold resonance) *Let $n = 1$. Then $H_\lambda^- f = 0$ has solution $f \in L^\varepsilon \setminus L^1$ for any $0 < \varepsilon < 1$. I.e., 0 is a super-threshold resonance of H_λ^-.*

Proof: $H_\lambda^- f = 0$ yields that

$$f(p) = \frac{\lambda}{2\pi} \frac{\sin p}{E(p)} \int_{\mathbb{T}} \sin p f(p) dp.$$

Note that however $\sin p / E(p) \notin L^1$, but we can see that $\sin p / E(p) \in L^\varepsilon$ for any $0 < \varepsilon < 1$ since $\sin p / E(p) \sim 1/p$ near $p = 0$ and $\int_{\mathbb{T}} p^{-\varepsilon} dp < \infty$. **qed**

Lemma 3.6 *(1) Let $n = 1$. Then 0 is neither a threshold resonance and a threshold eigenvalue, but for (λ, μ) with $\lambda \neq 0$, 0 is a super-threshold resonance.*

(2) Let $n = 2$. Then 0 is a threshold resonance at $\lambda = \lambda_s$.

(3) Let $n \geq 3$. Then 0 is a threshold eigenvalue at $\lambda = \lambda_s$ and its multiplicity is n.

Proof: (1) follows from Proposition 3.5. Let $n \geq 2$. Then the solution of $H_\lambda^- f = 0$ is given by (3.7). Since $\frac{\sin p_j}{E(p)} \in L^1 \setminus L^2$ for $n = 2$ and $\frac{\sin p_j}{E(p)} \in L^2$ for $n \geq 3$, we have $f \in L^1 \setminus L^2$ for $n = 2$, and $f \in L^2$ for $n \geq 3$. Then (2) and (3) follow. **qed**

4 Main theorems

4.1 Case of $n \geq 2$

In order to describe the main results we have to separate (λ, μ)-plane into several regions.

Lemma 4.1 *Let $n \geq 2$. Then $\lambda_\infty(z) \leq \lambda_s(z) \leq \lambda_c(z)$ for $z \in (-\infty, 0]$.*

Proof: It follows that $\lambda_c(z) = \frac{1}{c(z) - d(z)} > \lambda_s(z) = \frac{1}{s(z)} > \lambda_\infty(z) = \frac{a(z)}{b(z)}$ for $z < 0$. By a limiting argument the lemma follows. **qed**

Introduce four half planes:

$$\begin{aligned} \mathfrak{C}_- &= \{(\lambda, \mu) \in \mathbb{R}^2 | \lambda < \lambda_c\}, \quad \mathfrak{C}_+ = \{(\lambda, \mu) \in \mathbb{R}^2 | \lambda > \lambda_c\} \\ \mathfrak{S}_- &= \{(\lambda, \mu) \in \mathbb{R}^2 | \lambda < \lambda_s\}, \quad \mathfrak{S}_+ = \{(\lambda, \mu) \in \mathbb{R}^2 | \lambda > \lambda_s\}, \end{aligned}$$

and two vertical lines: $\beta_c = \{(\lambda, \mu) \in \mathbb{R}^2 | \lambda = \lambda_c\}$ and $\beta_s = \{(\lambda, \mu) \in \mathbb{R}^2 | \lambda = \lambda_s\}$. Note that $\mathfrak{S}_- \subset \mathfrak{C}_-$ and $\mathfrak{C}_+ \subset \mathfrak{S}_+$, and we define open sets by

$$\begin{aligned} &D_0 = G_0, \quad D_1 = G_1 \cap \mathfrak{S}_-, \quad D_2 = G_2 \cap \mathfrak{S}_-, \quad D_{n+1} = G_1 \cap (\mathfrak{S}_+ \cap \mathfrak{C}_-), \\ &D_{n+2} = G_2 \cap (\mathfrak{S}_+ \cap \mathfrak{C}_-), \quad D_{2n} = G_1 \cap \mathfrak{C}_+, \quad D_{2n+1} = G_2 \cap \mathfrak{C}_+. \end{aligned}$$

	D_0	D_1	D_2	D_{n+1}	D_{n+2}	D_{2n}	D_{2n+1}
E.v.in $(-\infty,0)$	0	1	2	$n+1$	$n+2$	$2n$	$2n+1$

	B_k		S_k		C_k		Point A		Point B	
E.v.$(-\infty,0)$	k		k		k		1		$n+1$	
0 res.	$n=2$	–	$n=2$	2	$n\geq 2$	–	$n=2$	2	$n=2$	–
	$n=3,4$	1	$n\geq 3$	–			$n=3,4$	1	$n=3,4$	1
	$n\geq 5$	–					$n\geq 5$	–	$n\geq 5$	–
0.e.v.	$n=2$	–	$n=2$	–	$n\geq 2$	$n-1$	$n=2$	–	$n=2$	1
	$n=3,4$	–	$n\geq 3$	n			$n=3,4$	n	$n=3,4$	$n-1$
	$n\geq 5$	1					$n\geq 5$	$n+1$	$n\geq 5$	n

Table 1: Spectrum of $H_{\lambda\mu}$ for (λ,μ) on D_k and the edges of D_k for $n\geq 2$

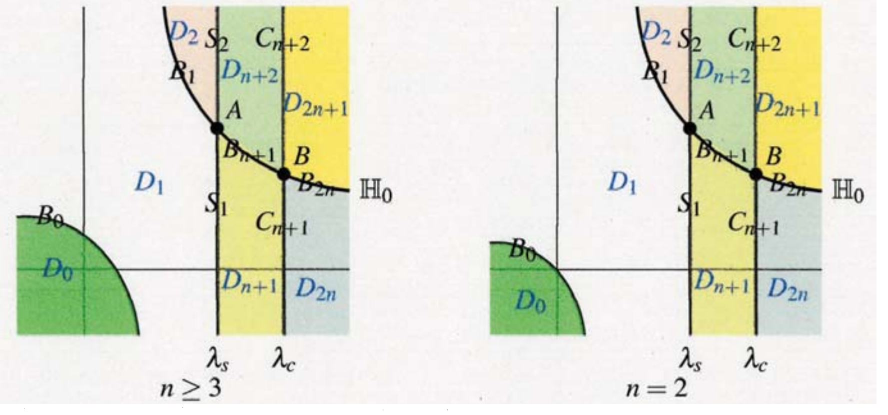


Figure 5: Hyperbola for $n\geq 2$

The boundaries of these sets define disjoint eight curves:

$$B_0=\beta_l,\quad B_1=\beta_r\cap\mathfrak{S}_-,\quad B_{n+1}=\beta_r\cap(\mathfrak{S}_+\cap\mathfrak{C}_-),\quad B_{2n}=\beta_r\cap\mathfrak{C}_+,$$
$$S_1=\beta_s\cap G_1,\quad S_2=\beta_s\cap G_2,\quad C_{n+1}=\beta_c\cap G_1,\quad C_{n+2}=\beta_c\cap G_2,$$

and two one point sets given by $A=\beta_r\cap\beta_s$ and $B=\beta_r\cap\beta_c$.

We are now in the position to state the main theorem for $n\geq 2$.

Theorem 4.2 *Let $n\geq 2$.*

(1) Assume that $(\lambda,\mu)\in D_k$, $k\in\{0,1,2,n+1,n+2,2n,2n+1\}$, then $H_{\lambda\mu}$ has k eigenvalues in $(-\infty,0)$. In addition $H_{\lambda\mu}$ has neither a threshold eigenvalue nor a threshold resonance.

(2) 0 is not a super-threshold resonance of $H_{\lambda\mu}$ for any $(\lambda,\mu)\in\mathbb{R}^2$.

(3) Assume that (λ,μ) in B_k, S_k, C_k and A,B results in Table 4.1 are true:

Proof: (1) follows from Lemmas 2.6, 2.9 and 3.3. (2) follows from Lemmas 2.13 3.4 and 3.6. (3) follows from Lemmas 2.15 and 3.6. **qed**

As is drawn in Figures 5 and 9, hyperbolas for $n = 1,2$ include the origin $(0,0)$, which is different from those for $n \geq 3$. It actually describes that the Hamiltonian $H_{\lambda\mu}$ with $\lambda = 0$ has a negative eigenvalue for any $\mu > 0$. This is clarified in e.g.,[3] and is also hold for one or two dimensional Schrödinger operator of the form $-(1/2)\Delta + V$. We refer to see e.g., [6] and reference therein. Moreover for any $(\lambda,\mu) \in (0,\infty) \times (0,\infty)$, $H_{\lambda\mu}$ always has negative eigenvalue and dose not have threshold resonance nor threshold eigenvalue for $n = 1,2$.

By a Rellich type theorem for the discrete Schödinger operator, embedded eigenvalues may be absent in the interval $(0,2n)$. Under some assumptions this is proven in [4]. We are however interested in constructing a Hamiltonian $H_{\lambda\mu}$ which has two eigenvalues on both edges of the spectrum $[0,2n]$. See Figure 6 below:

Figure 6: Two embedded eigenvalues or resonances of $H_{\lambda\mu}$

By the discussion stated in the previous sections we can construct an example of discrete Schrödinger operators which has two embedded eigenvalues. Let $n \geq 3$. Points (λ,μ) on the blue curve in the upper left construct Schrödinger operators $H_{\lambda\mu}$ which has resonance or eigenvalue at 0. On the other hand points (λ,μ) on the red curve in the upper right construct Schrödinger operators $H_{\lambda\mu}$ which has resonance or eigenvalue at $2n$. See Figure 7.

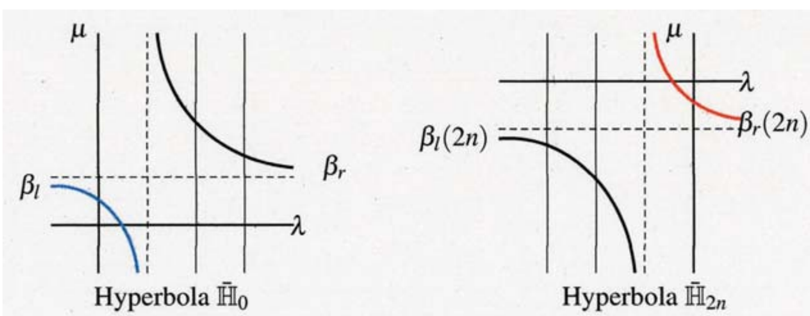


Figure 7: Resonance and eigenvalue on edges

We have the theorem.

Theorem 4.3 *(1) Let $n = 3,4$. Then there exists two pints (λ_1,μ_1) and (λ_2,μ_2) in (λ,μ)-plane such that both 0 and $2n$ of the spectrum $\sigma(H_{\lambda_j\mu_j})$ are simultaneously resonances. (2) Let $n \geq 5$. Then there exists two pints (λ_1,μ_1) and (λ_2,μ_2) in (λ,μ)-plane such that both 0 and $2n$ of the spectrum $\sigma(H_{\lambda_j\mu_j})$ are simultaneously eigenvalues.*

Proof: Two brunches $\beta_r(2n)$ and β_l cross at just two points (λ_1,μ_1) and (λ_2,μ_2) for $n \geq 3$, where $\lambda_1,\mu_2 < 0$ and $\lambda_2,\mu_1 > 0$. Refer to see Figure 8. In the case of $n = 3,4$, two points

0 and $2n$ are resonances and in the case of $n \geq 5$ these are eigenvalues. Then the proof is completed. **qed**

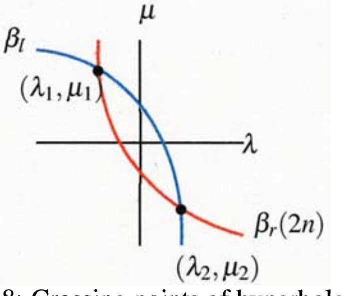


Figure 8: Crossing points of hyperbolas

4.2 Case of $n = 1$

Let $n = 1$. In this case, the asymptote of $\bar{\mathbb{H}}_0$ is $(\lambda_\infty(0), \mu_\infty(0)) = (1, 1)$, and λ_c is not defined. We also see that $\lambda_s = 1 = \lambda_\infty(0)$. Then we have 4 sets $D_0 = G_0, D_1 = G_1 \cap \mathfrak{S}_-, D_2 = G_1 \cap A_1$ and $D_3 = G_2$. The boundaries of these sets define disjoint three curves:$B_0 = \beta_l$, $B_2 = \beta_r$ and $A_1 = \beta_s$.

	D_0	D_1	D_2	D_3
E.v.in $(-\infty, 0)$	0	1	2	3

	B_k	A_1
E.v.in $(-\infty, 0)$	k	1
0 res.	–	–
0.e.v.	–	–

Table 2: Spectrum of $H_{\lambda\mu}$ for (λ, μ) on D_k and the edges of D_k for $n = 1$

Now we formulate next result for $n = 1$.

Theorem 4.4 *Let $n = 1$.*

(1) Assume $(\lambda, \mu) \in D_k, k \in \{0, 1, 2, 3\}$. Then $H_{\lambda\mu}$ has k eigenvalues in $(\infty, 0)$. In addition 0 is neither a threshold resonance and a threshold eigenvalue.

(2) Assume that (λ, μ) with $\lambda = \lambda_s = 1$. Then 0 is a super-threshold resonance.

(3) Assume that (λ, μ) in B_k or A_1. Then results in Table 4.2 are true: In particular $H_{\lambda\mu}$ has neither a threshold resonance nor a threshold eigenvalue.

Proof: The theorem follows from Lemmas 2.6, 2.9, 3.3. 2.15 and 3.6. **qed**

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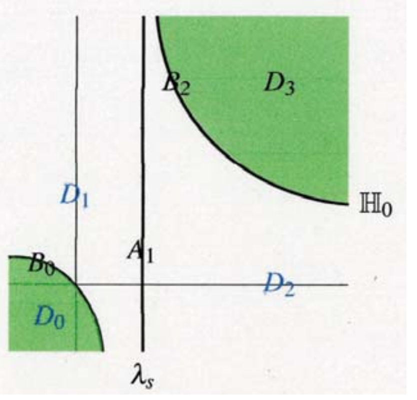

Figure 9: Hyperbola for $n = 1$

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